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ON SELF-SIMILAR SOLUTIONS OF THE SECOND KIND IN THE THEORY OF UNSTEADY FILTRATION

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If a self-similar solution is to be an asymptotic representation of a specific class of not self-similar motions, it must be stable with respect to small perturbations. Proof of the stability of self-similar solutions of the second kind of the Cauchy problem is given in linear approximation for the equation of elastic-plastic filtration mode derived in [1]. The solution of a similar axisymmetric problem is constructed.

1. As shown in [1], the self-similar solution of the Cauchy problem for one-dimensional equation of elastic-plastic filtration

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial u}{\partial t}\right)\frac{\partial^2 u}{\partial x^2}, \qquad a^2(z) = a_1^2 \ (z < 0); \qquad a_2^2 \ (z > 0) \tag{1.1}$$

is of the form

$$U_0 - u(x, t) = \frac{A}{(a_1^2 t)^{1/2(1+\alpha)}} f(\xi), \qquad \xi = \frac{x}{\sqrt{a_1^2 t}} \tag{1.2}$$

Here function f is expressed in terms of parabolic cylinder functions determined by the system of equations

$$D_{\alpha+2}(\xi_0 / \sqrt{2}) = 0, \quad M(-1 - {}^1/_2\alpha, {}^1/_2; {}^1/_4\xi_0^2\varepsilon^{-1}) = 0, \quad \varepsilon = a_2^2 / a_1^2 \tag{1.3}$$

with the exponent α and the value of $\xi = \xi_0$ such that $\partial u / \partial t = 0$ when $x = x_0(t) = \xi_0 \sqrt{a_1^2 t}$.

Let us consider the solution of a Cauchy problem with initial data defined at a certain instant t_0 are defined by the weakly perturbed self-similar solution

$$U_0 - u(x, t_0) = A\,[f(\xi) + \mu v(\xi, t_0)]\,(a_1^2 t_0)^{-1/2(1+\alpha)}$$

Here μ is small and function $v(\xi, t_0)$ is such that $u(x, t_0)$ and the self-similar solution have only two points of inflection. The surface on which $\partial u / \partial t = 0$ is, also, subject to perturbation and moves according to the law

$$x_1(t) = [\xi_0 + \beta_1(t)]\sqrt{a_1^2 t}, \qquad x_2(t) = -[\xi_0 + \beta_2(t)]\sqrt{a_1^2 t}$$

Substituting the perturbed solution into (1.1) and passing to variable $\tau = \ln t$, we obtain the equation

$$\frac{\partial v}{\partial \tau} = \varepsilon \frac{\partial^2 v}{\partial \xi^2} + \frac{\xi}{2}\frac{\partial v}{\partial \xi} + \frac{1+\alpha}{2} v \qquad (0 \leqslant \xi \leqslant \xi_0)$$

$$\frac{\partial v}{\partial \tau} = \varepsilon \frac{\partial^2 v}{\partial \xi^2} + \frac{\xi}{2}\frac{\partial v}{\partial \xi} + \frac{1+\alpha}{2} v + \frac{\varepsilon - 1}{\mu}\frac{d^2 f}{d\xi^2} \qquad (\xi_0 \leqslant \xi \leqslant \xi_0 + \beta_1(t)) \qquad (1.4)$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} + \frac{\xi}{2}\frac{\partial v}{\partial \xi} + \frac{1+\alpha}{2} v \qquad (\xi_0 + \beta_1(t) \leqslant \xi < \infty)$$

and similar equations for $\xi < 0$.

Linearizing the condition $\partial u / \partial t = 0$ at $\xi = \xi_0 + \beta_1(t)$, we find

$$\mu \left[\frac{\partial v}{\partial \tau} - \frac{1+\alpha}{2} v\right]_{\xi=\xi_0} - \frac{1+\alpha}{2} f'(\xi_0)\,\beta_1(t) = 0$$

Hence the shift of the boundary $\beta_1(t)$ is proportional to the value of the small perturbation. When $\xi = \xi_0$, the derivative $d^2 f / d\xi^2 = 0$, consequently the inhomogeneity in the intermediate region does not result in a discontinuity of $\partial v / \partial \xi$ for $\mu \to 0$. Linearizing (1.4), we obtain for v the linear equation

$$\frac{\partial v}{\partial \tau} = \varepsilon \frac{\partial^2 v}{\partial \xi^2} + \frac{\xi}{2}\frac{\partial v}{\partial \xi} + \frac{1+\alpha}{2} v \qquad (|\xi| \leqslant \xi_0)$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} + \frac{\xi}{2}\frac{\partial v}{\partial \xi} + \frac{1+\alpha}{2} v \qquad (|\xi| \leqslant \xi_0)$$

Here v and $\partial v / \partial \xi$ are continuous when $\xi = \xi_0$.

Let us assume that $v = \Sigma w_n(\xi) \exp(-{}^1/_2 \lambda_n \tau)$. To prove the stability it is necessary to show that all eigenvalues of λ are nonnegative and that with $t \to \infty$ the rate of decrease of perturbations of the self-similar solution is not slower than that of the solution itself. For function $w(\xi)$ we have the equation

$$\varepsilon \frac{d^2 w}{d\xi^2} + \frac{\xi}{2}\frac{dw}{d\xi} + \frac{1+\alpha+\lambda}{2} w = 0 \qquad (|\xi| \leqslant \xi_0) \qquad (1.5)$$

$$\frac{d^2 w}{d\xi^2} + \frac{\xi}{2}\frac{dw}{d\xi} + \frac{1+\alpha+\lambda}{2} w = 0 \qquad (|\xi| > \xi_0)$$

Here w and $dw / d\xi$ must be continuous for $\xi = \xi_0$. It is readily seen that the spectrum of this problem is discrete.

Within the limits of linearized theory it is sufficient to consider separately the symmetric and the skew-symmetric perturbations, w_1° and w_2°, respectively.

The symmetric solution of Eq. (1.5) satisfying the conditions at $\xi \to \infty$ is of the form

$$w_1^\circ = \begin{cases} C_1 \exp(-{}^1/_8 \xi^2 \varepsilon^{-1})\,[D_{\alpha+\lambda}(\xi/\sqrt{2\varepsilon}) + D_{\alpha+\lambda}(-\xi/\sqrt{2\varepsilon})], & |\xi| \leqslant \xi_0 \\ C_2 \exp(-{}^1/_8 \xi^2)\, D_{\alpha+\lambda}(|\xi|/\sqrt{2}), & |\xi| > \xi_0 \end{cases}$$

Here $D_\nu(z)$ is a parabolic cylinder function. The characteristic equation for λ, after passing to degenerated hypergeometric functions, reduces to

$$\Delta(\lambda) = (\alpha + \lambda + 1) D_{\alpha+\lambda}(\zeta_0) M(-1 - {}^1/_2(\alpha + \lambda), {}^1/_2; {}^1/_2\zeta_0^2\varepsilon^{-1}) + D_{\alpha+\lambda+2}(\zeta_0) M(-{}^1/_2(\alpha + \lambda), {}^1/_2; {}^1/_2\zeta_0^2\varepsilon^{-1}) = 0 \quad (1.6)$$

Here α and $\zeta_0 = \xi_0 / \sqrt{2}$ are defined by (1.3). It will be readily seen from (1.3) that $\lambda_0 = 0$ is a root of Eq. (1.6). We shall prove that its remaining roots are positive. We transform Eq. (1.6) to the form

$$\begin{aligned}\Delta(\lambda) = \zeta_0 D_{\alpha+\lambda+1}(\zeta_0) M(-1 - {}^1/_2(\alpha + \lambda), {}^1/_2; {}^1/_2\zeta_0^2\varepsilon^{-1}) + \\ + D_{\alpha+\lambda+2}(\zeta_0) [M(-{}^1/_2(\alpha + \lambda), {}^1/_2; {}^1/_2\zeta_0^2\varepsilon^{-1}) - \\ - M(-1 - {}^1/_2(\alpha + \lambda), {}^1/_2; {}^1/_2\zeta_0^2\varepsilon^{-1})] = 0\end{aligned} \quad (1.7)$$

According to [2], $M(a + l, {}^1/_2, x_0)$ is a monotonically increasing function of l when $l > 0$, and x_0 is the smallest positive root of equation $M(a, {}^1/_2, x) = 0$. If ζ_0 is the smallest positive root of equation $D_{\alpha+2}(\zeta) = 0$, then $D_{\alpha+\lambda+2}(\zeta_0) > 0$ for $\lambda < 0$. Hence $\Delta(\lambda) > 0$ for $\lambda < 0$.

The analogous skew-symmetric solution is of the form

$$w_2^\circ = \begin{cases} C_3 \exp(-{}^1/_8\xi^2\varepsilon^{-1}) [D_{\alpha+\lambda}(\xi / \sqrt{2\varepsilon}) - D_{\alpha+\lambda}(-\xi / \sqrt{2\varepsilon})], & 0 \leqslant \xi < \xi_0 \\ C_4 \exp(-{}^1/_8\xi^2) D_{\alpha+\lambda}(\xi / \sqrt{2}), & \xi_0 \leqslant \xi < \infty \end{cases} \quad (1.8)$$

The characteristic equation is of the form

$$\begin{aligned}\Delta_1(\lambda) = \zeta_0 D_{\alpha+\lambda}(\zeta_0) M(-1 - {}^1/_2(\alpha + \lambda - 1), {}^1/_2, {}^1/_2\zeta_0^2\varepsilon^{-1}) + \\ + \varepsilon D_{\alpha+\lambda+1}(\zeta_0) [M(-{}^1/_2(\alpha + \lambda - 1), {}^1/_2; {}^1/_2\zeta_0^2\varepsilon^{-1}) - \\ - M(-1 - {}^1/_2(\alpha + \lambda - 1), {}^1/_2, {}^1/_2\zeta_0^2\varepsilon^{-1})] = 0\end{aligned} \quad (1.9)$$

Comparison of this equation with (1.7) shows that the smallest root of Eq. (1.9) is $\lambda_1 = 1$. Hence the self-similar solution is stable with respect to small perturbations.

The analysis shows that $\lambda_2 = 2$ is the smallest positive root of Eq. (1.6), and $\lambda_3 = 3$ is the corresponding root of Eq. (1.9). For considerable periods of time the asymptotic solution may be presented in the form

$$U_0 - u(x, t) = \frac{A}{(a_1^2 t)^{1/2(1+\alpha)}} \left[(1 + c_0) f(\xi) + \frac{c_1}{t^{1/2}} w_1(\xi) + \frac{c_2}{t} w_2(\xi) + \frac{c_3}{t^{3/2}} w_3(\xi) + o\left(\frac{1}{t^{3/2}}\right) \right] \quad (1.10)$$

as was done in [3].

The constant c_0 is, generally speaking, nonzero, and this confirms the indeterminacy of constant A in the self-similar statement of the problem in [1].

2. The stability of a self-similar solution of the dipole kind is analyzed in a similar manner [1]. The perturbable solution is of the form (1.2), where α and ξ_0 are determined from the system

$$D_{\alpha+2}(\zeta_0) = 0, \quad M(-{}^1/_2\alpha, {}^3/_2; {}^1/_2; \zeta^2{}_0\,\varepsilon^{-1}) = 0$$

Perturbation $v(\xi, t)$ must vanish at $\xi = 0$, and the eigenfunctions $w(\xi)$ are then defined by expression (1.8). The characteristic equation is of the form

$$\begin{aligned}(\alpha + \lambda) D_{\alpha+\lambda}(\zeta_0) M(-{}^1/_2(\alpha + \lambda + 1), {}^3/_2; {}^1/_2\zeta^2{}_0\varepsilon^{-1}) + \\ + D_{\alpha+\lambda+2}(\zeta_0) M(1 - {}^1/_2(\alpha + \lambda + 1), {}^3/_2; {}^1/_2\zeta^2{}_0\varepsilon^{-1}) = 0\end{aligned}$$

The smallest root of this equation is $\lambda = 0$, i.e. the dipole type solution is stable.

3. Let us construct by the method proposed in [1] a self-similar solution of the problem of elastic-plastic filtration from a momentary axisymmetric source.

For the equation of the elastic-plastic filtration mode

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial u}{\partial t}\right)\Delta u \tag{3.1}$$

we formulate the nonself-similar Cauchy problem with axisymmetric input data

$$U_0 - u(\rho, 0) = \frac{Q}{2\pi R^2} u_0\left(\frac{\rho}{R}\right), \qquad \int_0^\infty u_0(r)\, r dr = 1 \tag{3.2}$$

(ρ is a cylindrical coordinate).

Dimensional analysis makes it possible to present the solution in the form

$$U_0 - u(\rho, t) = \frac{Q}{a_1^2 t} F(\xi, \eta; \varepsilon), \quad \xi = \frac{\rho}{\sqrt{a_1^2 t}}, \quad \eta = \frac{R}{\sqrt{a_1^2 t}}, \quad \varepsilon = \frac{a_2^2}{a_1^2}$$

The assumption that for η tending to zero function F has finite asymptotics is valid only in the linear case $\varepsilon = 1$. It may be generally assumed that there is an α such that for $\eta \to 0$ there exists $\lim [\eta^{-\alpha} F(\xi, \eta; \varepsilon)] = f(\xi; \varepsilon)$. The self-similar solution of the problem (3.1), (3.2) is then sought in the form

$$U_0 - u(\rho, t) = \frac{A}{(a_1^2 t)^{1+1/2\alpha}} f(\xi; \varepsilon) \tag{3.3}$$

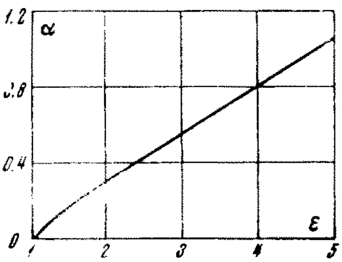

Fig. 1

The equation of the function assumes the form

$$\varepsilon \frac{d^2 f}{d\xi^2} + \left(\frac{\varepsilon}{\xi} + \frac{\xi}{2}\right)\frac{df}{d\xi} + \frac{2+\alpha}{2} f = 0 \quad (0 \leqslant \xi \leqslant \xi_0) \tag{3.4}$$

$$\frac{d^2 f}{d\xi^2} + \left(\frac{1}{\xi} + \frac{\xi}{2}\right)\frac{df}{d\xi} + \frac{2+\alpha}{2} f = 0 \quad (\xi_0 \leqslant \xi < \infty)$$

The solution of Eq. (3.4), which is regular for $\xi = 0$ and satisfies the condition

$$\int_0^\infty f(\xi)\, \xi^{\alpha+1} d\xi = \frac{1}{2\pi}$$

is expressed in terms of hypergeometric functions M and U [2]

$$f(\xi) = \begin{cases} B \exp(-\tfrac{1}{4}\xi^2\varepsilon^{-1})\, M(-\tfrac{1}{2}\alpha, 1; \tfrac{1}{4}\xi^2\varepsilon^{-1}), & 0 \leqslant \xi \leqslant \xi_0 \\ C \exp(-\tfrac{1}{4}\xi^2)\, U(-\tfrac{1}{2}\alpha, 1; \tfrac{1}{4}\xi^2), & \xi_0 \leqslant \xi < \infty \end{cases}$$

The condition that $\partial u / \partial t = 0$ at $\rho = \xi_0 \sqrt{a_1^2 t}$ yields for the determination of α and ξ_0 the system of transcedental equations

$$M(-\tfrac{1}{2}\alpha, 2; z_0\varepsilon^{-1}) - 2M(-1 - \tfrac{1}{2}\alpha, 1; z_0\varepsilon^{-1}) = 0$$

$$U(-\tfrac{1}{2}\alpha, 2; z_0) + 2U(-1 - \tfrac{1}{2}\alpha, 1; z_0) = 0 \qquad (z_0 = \tfrac{1}{4}\xi_0^2)$$

Values of α calculated for several ε are given below

$\varepsilon =$	1.1	1.2	1.5	2.0	3.0	4.0
$\alpha =$	0.03	0.065	0.16	0.32	0.57	0.81

The pattern of the dependence $\alpha(\varepsilon)$ is shown in Fig. 1. For $\varepsilon = 1$ we have $\alpha = 0$, thus yielding the conventional solution of Poisson. For $\varepsilon > 1$ we obtain, as in the one-dimensional case, $\alpha > 0$.

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